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Mean Field Behaviour of Spin Systems with Orthogonal Interaction Matrix

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For the long-range deterministic spin models with glassy behaviour of Marinari, Parisi and Ritort we prove weighted factorization properties of the correlation functions which represent the natural generalization of the factorization rules valid for the Curie–Weiss case.

KEY WORDS: Deterministic spin glasses, mean field, factorization rules.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Mean field models in statistical mechanics are often introduced to provide a simplification of other more realistic ones. Their success is based upon the robust physical meaning of the involved approximation: each part of the system is considered to feel the action of the remaining ones through a mean effect which decreases with the total size of the system. The notion of finite cubes immersed in a d-dimensional lattice with Euclidean distance is then replaced by the complete graph plunged in an infinite dimensional lattice whose distance among points decreases uniformly with the size of the graph.

The simplest and most celebrated among those models is the Curie–Weiss one: the first microscopic theory of ferromagnetism was built on its exact solution. The expression *exact solution* has here a peculiarly strong meaning: not only the free energy density can be computed in closed form in the thermodynamic limit but also the entire family of correlation functions. In fact it turns out that once the two point correlation function is

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known all the higher order correlations can be computed as powers of the former after the thermodynamic limit is performed (actually the computations yields even order correlation functions; the odd ones are all zero by symmetry in the zero external magnetic field case). The theory is said to have an order parameter, in this case the local magnetization, and the factorization property of the correlation functions can be considered the mathematical description of a *mean field behaviour*.

In this paper we study the sine model defined by the Hamiltonian

$$\mathscr{H}_N(\sigma) = -\sum_{i<j} \frac{1}{\sqrt{2N+1}} \sin\left(\frac{2\pi ij}{2N+1}\right) \sigma_i \sigma_j \tag{1.1}$$

or more generally a spin system with orthogonal interaction matrix.

This class of models has been introduced by Marinari, Parisi and Ritort in ref. 1 and subsequently studied in ref. 2. In the sequel we shall refer to them as MPR models.

They probably provide the first example of long range spin models with non-random interactions with a genuine mean field spin-glass low-temperature phase. On the other hand the Hamiltonian (1.1) shares with the Sherrington Kirkpatrick one a mean field property since the interaction felt by each spin due to the remaining ones is in the average the same. In SK the local field is a Gaussian variable with zero average and unit variance. In this case the local field

$$h_i = \sum_{j} \frac{1}{\sqrt{2N+1}} \sin\left(\frac{2\pi ij}{2N+1}\right) \sigma_j \tag{1.2}$$

can be considered in a natural way as a random variable uniformly distributed over the lattice $\mathbf{Z}_N := \mathbf{Z} \bmod N$ with zero average and unit variance.

Our main objective is to establish to what extent the mean field property of the Hamiltonian is reflected on the factorization properties of the correlation functions.

The main result of the paper is the natural generalization of the factorization rule valid for the Curie–Weiss case and can be expressed by the following

Proposition 1.1. For every positive inverse temperature β except, at most, a set of zero Lebesgue measure, and in particular for every $\beta < 1$ the correlation functions fullfill the following relation:

$$\left| \frac{1}{N^2} \sum J_{i,j} J_{l,m} \langle \sigma_i \sigma_j \sigma_l \sigma_m \rangle - \frac{1}{N^2} \sum J_{i,j} J_{l,m} \langle \sigma_i \sigma_j \rangle \langle \sigma_l \sigma_m \rangle \right| = \mathscr{O}\left(\frac{1}{N}\right)$$

where the sums run over non-coincident indices within each expectation.

Remarks.

1. A little algebra after application of translation invariance shows that the previous formula, when the interaction coefficients are $\frac{1}{N}$ (Curie–Weiss) yields the well known factorization rule $\langle \sigma_1\sigma_2\sigma_3\sigma_4 \rangle = \langle \sigma_1\sigma_2 \rangle^2$ in the thermodynamic limit.

2. Higher order relations can be found involving pairwise n-points connected correlations, with $n \geqslant 2$. We postpone a more precise statement of these results to Sections 2 and 3.

3. It is perhaps worth mentioning that such an even-type factorization property is structurally different from the factorization property of pure states (see, e.g., ref. 3, III.1): the first one describes the reduction of the Gibbs state to the two point correlation function like in the Gaussian case, while the latter doesn't hold for all the Gibbs states but only for the extremal ones in which the former can be decomposed. In each of them there is a complete factorization of the correlation functions when the thermodynamic limit is performed.

Our strategy is the following: from the study of the fluctuations of the intensive quantities (basically the energy per particle) we deduce factorization properties for the correlation functions for our non translationally-invariant interactions. Similar results were obtained in refs. 4 and 5 for SK. We want to stress the fact that our approach doesn't rely on the computation of the solution of the model (still not available at least on rigorous grounds) but only on those bounds over the fluctuation of the energy coming from equivalence of ensembles (microcanonical and canonical) ideas. The main technical tool we use is the property of orthogonality of the interaction matrix: it allows us to show first the extensivity of the energy and second to produce the expected $1/N$ bound on the fluctuations.

The paper is organized as follows: in the coming Section 2 we review the factorization properties of the Curie–Weiss model through an analysis of the energy fluctuations and of the high temperature expansion. We emphasize the fact that all the results we present are obtained, including the existence of the thermodynamical limit, without making use of the exact solution of the model. In Section 3 we apply the same methods to the MPR models and we obtain a properly weighted factorization formula.

2. REMARKS ON THE CURIE–WEISS MODEL

We begin this paper with a full description of the high-temperature ($\beta < 1$) regime of the Curie–Weiss model of statistical mechanics along with a discussion of the factorization properties of the correlation functions which can be obtained in this regime.

The basic setup is a probability space $(\Sigma_N, \mathscr{F}_N, \mathrm{P}_N)$ defined as follows: the sample space Σ_N is the configuration space, i.e., $\Sigma_N = \{-1, 1\}^N$ whose elements are sequences $\sigma = \sigma_1 \cdots \sigma_N$ such that $\sigma_i \in \{-1, 1\}$, $\mathscr{F}_N$ is the finite algebra with 2^{2^N} elements and the *a priori* (or *infinite-temperature*) probability measure P_N is given by

$$\mathrm{P}_N(C) = \frac{1}{2^N} \sum_{\sigma \in C} 1 \tag{2.1}$$

We shall consider systems specified by a global pair interaction Hamiltonian

$$\mathscr{H}_N(\sigma) = -\sum_{1 \leqslant i < j \leqslant N} J_{ij} \sigma_i \sigma_j \tag{2.2}$$

where $J = (J_{ij})$ is a symmetric nonnegative definite $N \times N$ matrix given from the outset.

The simplest example is the so called Curie–Weiss model, defined by $J_{ij} \equiv 1/N$. The partition function Z_N at inverse temperature β is defined as

$$Z_N(\beta) = \sum_{\sigma \in \Sigma_N} \exp(-\beta \mathscr{H}_N(\sigma)) = 2^N \, \mathrm{E}_N(e^{-\beta \mathscr{H}_N}) \tag{2.3}$$

The Hamiltonian for the Curie–Weiss model is then given by

$$\mathscr{H}_N(\sigma) = -\frac{1}{2N} \left(\sum_i \sigma_i \right)^2 + \frac{1}{2} = -\frac{1}{2N} \mathscr{M}_N^2(\sigma) + \frac{1}{2} \tag{2.4}$$

where

$$\mathscr{M}_N(\sigma) = \sum_{i=1}^{N} \sigma_i \tag{2.5}$$

is the total magnetization and the $1/2$ comes from the fact that we are not allowing self-interactions (there are no terms with $i = j$ in (2.2)). In particular we have the bounds

$$-\tfrac{1}{2}(N-1) \leqslant \mathscr{H}_N(\sigma) \leqslant \tfrac{1}{2} \tag{2.6}$$

The ground state σ^0 is the state which maximizes the magnetization, i.e., $\mathscr{M}_N(\sigma^0) = N$.

An important yet not well known fact about the Curie–Weiss model is that the existence of its thermodynamical limit can be proved, as a consequence of the subadditivity of the free energy density, independently of its exact solution and using only the bounds on the energy.

Proposition 2.1. For every positive integer k

$$\frac{1}{kN}\log Z_{kN} \leqslant \frac{1}{N}\log Z_N \tag{2.7}$$

Proof. We show the formula for $k=2$; the general case runs identically. The main ingredient is a lemma that can be proved by an easy combinatorial counting argument:

Lemma 2.1. Let P_N be the number of ways in which the set of $2N$ indices $1, 2,\ldots, 2N$ can be split into two sets of N indices and denote $\mathscr{P}_N$ the set of bipartitions, $P_N = |\mathscr{P}_N|$. Then the following identity holds:

$$\mathscr{H}_{2N} = \frac{1}{P_N} \sum_{p\in\mathscr{P}_N} (\mathscr{H}^{l}_{N}(p) + \mathscr{H}^{r}_{N}(p)) \tag{2.8}$$

where l and r stand for the left and right side of the bipartition p.

Introducing the uniform probability measure $\mathscr{E}$ on $\mathscr{P}_N$ and using the Jensen inequality we may apply the Griffiths symmetrization argument to get

$$\begin{aligned} Z_{2N} &= \sum \exp^{-\beta\mathscr{H}_{2N}} \\ &= \sum \exp^{-\beta\mathscr{E}(\mathscr{H}^{l}_{N}+\mathscr{H}^{r}_{N})} \\ &\leqslant \sum \mathscr{E}[\exp^{-\beta(\mathscr{H}^{l}_{N}+\mathscr{H}^{r}_{N})}] \\ &= \mathscr{E}(Z_N^2) = Z_N^2 \quad \blacksquare \end{aligned} \tag{2.9}$$

We show now how the theory of equivalence of ensembles can be used to prove a factorization property of the 4-points correlation function. From the equivalence of the canonical and the microcanonical ensemble we know that the energy density of the Curie–Weiss model $h_N = \mathscr{H}_N/N$ has vanishing fluctuations in the Gibbs state when the thermodynamic limit is performed.

In particular the quadratic fluctuations for almost all β (but for a set of zero Lebesgue measure) are ruled by:

$$\langle h_N^2 \rangle - \langle h_N \rangle^2 = \mathcal{O}\left(\frac{1}{N}\right) \tag{2.10}$$

An easy computation using the explicit form of the Hamiltonian shows that the former relation becomes, once the limit $N \to \infty$ is considered and the translation invariance has been taken into account (see also below):

$$\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle = \langle \sigma_1 \sigma_2 \rangle^2 \tag{2.11}$$

which expresses the vanishing of the 4-points truncated correlations.

This result comes from general facts about the Gibbs state and holds for almost all temperatures. On the other hand it can be improved to hold for all high temperatures, where one can also get similar results for more general $2n$-points truncated correlations, with $n \geqslant 2$.

To this purpose we can compute the partition function on that regime (see, e.g., ref. 6) by first decoupling the spins in the Hamiltonian through the elementary identity:

$$e^{a^2 b} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2} + \sqrt{2b}\, ax\right) dx \tag{2.12}$$

with the identifications $a = \mathcal{M}_N(\sigma)$ and $b = \beta/2N$. Since

$$\sum_{\sigma \in \Sigma_N} \exp\left(\sqrt{\frac{\beta}{N}}\, \mathcal{M}_N(\sigma)\, x\right) = 2^N \left[\cosh\left(\sqrt{\frac{\beta}{N}}\, x\right)\right]^N \tag{2.13}$$

we obtain

$$\begin{aligned} Z_N(\beta) &= 2^N \frac{e^{-\frac{\beta}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \left[\cosh\left(\sqrt{\frac{\beta}{N}}\, x\right)\right]^N dx \\ &= 2^N e^{-\frac{\beta}{2}} \sqrt{\frac{N}{2\pi\beta}} \int_{-\infty}^{\infty} \exp\left\{N\left[-\frac{y^2}{2\beta} + \log\cosh y\right]\right\} dy \end{aligned} \tag{2.14}$$

This formula immediately leads to the following result.

Proposition 2.2. For $0 \leqslant \beta < 1$ we have

$$-\beta F(\beta) \equiv \log Z_N(\beta) = N \log 2 + G_N(\beta) \tag{2.15}$$

where

$$G_N(\beta) \nearrow \sum_{k \geqslant 2} \frac{\beta^k}{2k} = -\frac{\beta}{2} - \log \sqrt{1-\beta} \qquad \text{as} \quad N \to \infty \tag{2.16}$$

Remark 1. Note that if one includes self-interactions in (2.2), namely for an Hamiltonian $\mathscr{H}_N(\sigma) = -2N^{-1}(\sum_i \sigma_i)^2$, the resulting limiting function is just $-\log \sqrt{1-\beta}$.

Proof. Using (2.13) with $(\cosh x)^N = \sum_{k=0}^{\infty} x^{2k} \sum_{k_1+\cdots+k_N=k} \prod_{l=1}^{N} \frac{1}{(2k_l)!}$ and observing that the combinatorial factor is $\binom{N}{k}/(2^k N^k) = 1/(2^k k!) + \mathscr{O}(\frac{1}{N})$ we get (with two successive change of variables $u^2 = 2x$ and $y = (1-\beta)\, x$, and $\beta < 1$)

$$Z_N(\beta) = 2^N\, e^{-\frac{\beta}{2}} \sqrt{\frac{2}{2\pi}} \int_0^{\infty} e^{-(x+\frac{1}{2})} \sum_0^{\infty} \frac{\beta^k}{k!}\, x^k x^{-\frac{1}{2}}\, dx + \mathscr{O}\left(\frac{1}{N}\right) \tag{2.17}$$

$$= 2^N \frac{e^{-\frac{\beta}{2}}}{\sqrt{1-\beta}} \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-y} y^{-\frac{1}{2}}\, dy + \mathscr{O}\left(\frac{1}{N}\right) \tag{2.18}$$

which gives the theorem with the observation $\Gamma(0) = \sqrt{\pi}$. ∎

We will now deal with the limiting behaviour of the energy density and connected correlations at high temperature deriving some easy consequences of the above result. For notational simplicity' sake let $\langle \cdot \rangle$ denote the thermal average corresponding to fixed β and N[3], i.e., given $A\colon \Sigma_N \to \mathbb{R}$,

$$\langle A \rangle = \frac{\sum_{\sigma \in \Sigma_N} A(\sigma) \exp(-\beta \mathscr{H}_N(\sigma))}{Z_N(\beta)} \tag{2.19}$$

Define moreover the energy density

$$h_N(\sigma) = \frac{\mathscr{H}_N(\sigma)}{N} \tag{2.20}$$

which, by (2.6), takes values in the interval $[-\frac{1}{2}, \frac{1}{2N}]$, and consider its distribution function at (inverse) temperature β,

$$F_N(t) = \langle \chi_{\{h_N \leqslant t\}} \rangle \tag{2.21}$$

[3] A more consistent notation would be $\mathrm{E}_{N,\beta}(\cdot)$, so that $\mathrm{E}_N(\cdot) \equiv \mathrm{E}_{N,0}(\cdot)$.

The following expression for the Laplace transform, or characteristic function, of F_N is easily obtained from (2.3) and (2.19):

$$\varphi_N(\lambda)=\int e^{-\lambda t}\,dF_N(t)=\langle e^{-\lambda h_N}\rangle=\frac{Z_N\left(\beta+\frac{\lambda}{N}\right)}{Z_N(\beta)} \tag{2.22}$$

On the other hand, by Proposition 2.2 and the mean value theorem we have that, for $0\leqslant\beta<1$ and N large enough

$$\frac{Z_N\left(\beta+\frac{\lambda}{N}\right)}{Z_N(\beta)}=\exp\left(\frac{\lambda}{N}\,G'_N(\beta^*)\right)=1+\mathcal{O}\left(\frac{\lambda}{N}\right) \tag{2.23}$$

for some β^* such that $0\leqslant\beta^*-\beta\leqslant\lambda/N$. We may now use a well known theorem of probability theory (see, e.g., ref. 7) which says that F_N converges weakly to F if and only if $\varphi_N(\lambda)\to\varphi(\lambda)$ for any λ (where $\varphi(\lambda)$ is the characteristic function of F). Noting that $\varphi(\lambda)=1$ is the characteristic function of the distribution function $G(t)=\chi_{[0,\infty)}(t)$ we have obtained the following result

Proposition 2.3. For $0\leqslant\beta<1$ and $N\to\infty$ the energy densities h_N converge in distribution to a random variable h which is δ-distributed at $x=0$.

Moreover, since the range of h is the interval $[-1/2, 1/2N]$, the random variables h_N^n are uniformly integrable for each $n\in\mathbb{N}$, i.e., for some $\epsilon>0$

$$\sup_N\langle|h_N|^{n+\epsilon}\rangle<\infty \tag{2.24}$$

By another well known theorem of probability theory (see ref. 7) the bound (2.24) along with Proposition 2.3 imply that

$$\langle h_N^n\rangle\to\langle h^n\rangle,\qquad N\to\infty,\quad n\in\mathbb{N} \tag{2.25}$$

But we can say more. Indeed, by virtue of (2.24) the expansion of the function $\varphi_N(\lambda)$ in powers of λ, i.e.

$$\varphi_N(\lambda)=\sum_{n=0}^{\infty}(-1)^n\frac{\lambda^n}{n!}\langle h_N^n\rangle \tag{2.26}$$

converges absolutely in a domain of the complex λ-plane which contains the point $\lambda = 0$ and can be taken independent of N. Therefore, a standard Cauchy-type estimate along with (2.23) yield

$$|\langle h_N^n \rangle - \langle h^n \rangle| = \mathcal{O}\left(\frac{C}{N}\right) \tag{2.27}$$

where C is a positive constant depending on n but not on N. On the other hand we have, as $N \to \infty$,

$$-\langle h_N \rangle = \frac{1}{N^2} \sum_{i<j} \langle \sigma_i \sigma_j \rangle = \frac{N(N-1)}{2N^2} \langle \sigma_1 \sigma_2 \rangle \to \frac{\langle \sigma_1 \sigma_2 \rangle}{2} \tag{2.28}$$

Moreover, when computing

$$\langle h_N^2 \rangle = \frac{1}{N^4} \sum_{i<j,\, l<k} \langle \sigma_i \sigma_j \sigma_l \sigma_k \rangle$$

the only terms which survive in the limit $N \to \infty$ are those having all the indices distinct. These can be further divided into six classes corresponding to the ordered quadruples $i<j<l<k$, $i<l<j<k$, $i<l<k<j$ plus those obtained by interchanging i with l and j with k. Each class contains $N(N-1)(N-2)(N-3)/4!$ terms. Hence we get

$$\langle h_N^2 \rangle \to \frac{\langle \sigma_1 \sigma_2 \sigma_3 \sigma_4 \rangle}{4}, \qquad N \to \infty \tag{2.29}$$

Now by Proposition 2.3 we have that $\langle h \rangle = 0$ and $\mathrm{Var}\, h = \langle h^2 \rangle - \langle h \rangle^2 = 0$. Putting together these facts and (2.25), (2.28), (2.29) we recover the factorization rule (2.11). Notice that we can write

$$\langle h_N \rangle = -\frac{1}{N}\left(\frac{\partial \log Z_N}{\partial \beta}\right) = -\frac{1}{N^2} \sum_{i<j} \langle \sigma_i \sigma_j \rangle$$

Similarly,

$$\begin{aligned} \mathrm{Var}\, h_N = \langle h_N^2 \rangle - \langle h_N \rangle^2 &= \frac{1}{N^2}\left(\frac{\partial^2 \log Z_N}{\partial \beta^2}\right) \\ &= \frac{1}{N^4} \sum_{i<j,\, l<k} \langle \sigma_i \sigma_j, \sigma_l \sigma_k \rangle_c \end{aligned}$$

where

$$\langle \sigma_i \sigma_j, \sigma_l \sigma_k \rangle_c := \langle \sigma_i \sigma_j \sigma_l \sigma_k \rangle - \langle \sigma_i \sigma_j \rangle \langle \sigma_l \sigma_k \rangle \tag{2.30}$$

More generally, for $n < N/2$ we can write the nth *moment* of h_N as

$$\begin{aligned}\langle h_N^n \rangle &= \frac{(-1)^n}{Z_N\, N^n} \left(\frac{\partial^n Z_N}{\partial \beta^n} \right) \\ &= \frac{(-1)^n}{N^{2n}} \sum_{i_n < j_1, \ldots, i_n < j_n} \langle \sigma_{i_n} \sigma_{j_1} \cdots \sigma_{i_n} \sigma_{j_n} \rangle \end{aligned} \tag{2.31}$$

The nth *cumulant* is then defined as

$$\begin{aligned}\langle h_N^n \rangle_c &:= \frac{(-1)^n}{N^n} \left(\frac{\partial^n \log Z_N}{\partial \beta^n} \right) \\ &= \frac{(-1)^n}{N^{2n}} \sum_{i_n < j_1, \ldots, i_n < j_n} \langle \sigma_{i_n} \sigma_{j_1}, \ldots, \sigma_{i_n} \sigma_{j_n} \rangle_c \end{aligned} \tag{2.32}$$

where $\langle \sigma_{i_n} \sigma_{j_1}, \ldots, \sigma_{i_n} \sigma_{j_n} \rangle_c$ is called the *pairwise n-points connected* (or *truncated*) correlation, and is defined recursively by

$$\begin{aligned}&\langle \sigma_{i_n} \sigma_{j_1}, \ldots, \sigma_{i_n} \sigma_{j_n} \rangle_c \\ &\quad = \langle \sigma_{i_n} \sigma_{j_1} \cdots \sigma_{i_n} \sigma_{j_n} \rangle - \sum_{\substack{\text{partitions of} \\ i_n j_1, \ldots, i_n j_n}} \left[\begin{array}{c} \text{products of pairwise } m\text{-points} \\ \text{connected correlations with } m < n \end{array} \right] \end{aligned} \tag{2.33}$$

While the moments $\langle h_N^n \rangle$ are somehow redundant in that they carry information on correlations among k spins with $k \leqslant n$ (so that part of this information is already stored in lower order moments), the cumulants $\langle h_N^n \rangle_c$ carry only the new information concerning n spins.

Using once more the fact that h_N^n is uniformly integrable, Proposition 2.3 and (2.27) we see that $\langle h_N^n \rangle_c = \mathcal{O}(C\, N^{-1})$ for each fixed $n \in \mathbb{N}$ and $N \to \infty$.

We have therefore proved the following

Proposition 2.4. For $0 \leqslant \beta < 1$ and for each $n > 1$ we can find a positive constant $C = C(n)$ so that

$$\langle \sigma_{i_n} \sigma_{j_1}, \ldots, \sigma_{i_n} \sigma_{j_n} \rangle_c = \mathcal{O} \left(\frac{C}{N} \right) \tag{2.34}$$

as $N \to \infty$.

Remark 2. By a straightforward inductive argument based on the recursive formula (2.33) and on translation invariance it is easily seen that in the thermodynamic limit (2.34) amounts to the simple factorization property

$$\langle \sigma_1 \sigma_2 \cdots \sigma_{2n-1} \sigma_{2n} \rangle = \langle \sigma_1 \sigma_2 \rangle^n \tag{2.35}$$

3. THE ORTHOGONAL MODEL

We shall now extend the results obtained in the previous section to the more interesting class of (non-translationally invariant) interactions

$$\mathscr{H}_N(\sigma) = -\tfrac{1}{2} \sum_{i,j \in \mathbf{Z}_N^2} J_{ij}\, \sigma_i\, \sigma_j \tag{3.1}$$

where $\mathbf{Z}_N$ is the integer lattice $\mathbf{Z}_N = \mathbf{Z} \bmod N$ and $J = (J_{ij})$ is a symmetric real orthogonal $N \times N$ matrix. This means that J has the form $J = OLO^T$ with L a diagonal matrix with elements ± 1 and O a generic orthogonal matrix chosen at random w.r.t. the Haar measure on the orthogonal group. The knowledge of the eigenvalues of J imposes simple bounds on the energy of any spin configuration. Here, due to orthogonality, the possible eigenvalues are $+1$, -1 so that

$$-\frac{N}{2} \leqslant \mathscr{H}_N(\sigma) \leqslant \frac{N}{2} \tag{3.2}$$

An important example for our purposes is given by the *sine model* where

$$J_{i,j} = \frac{2}{\sqrt{2N+1}} \sin\left(\frac{2\pi i j}{2N+1}\right) \tag{3.3}$$

which satisfies (see, e.g., refs. 1 and 2):

$$JJ^T = \mathrm{Id} \tag{3.4}$$

$$\sum_{i=1}^{N} J_{ii} = 1 \qquad \text{for } N \text{ odd;} \tag{3.5}$$

$$\sum_{i=1}^{N} J_{ii} = 0 \qquad \text{for } N \text{ even;} \tag{3.6}$$

$$\sum_{i=1}^{N} J_{ii}^2 = 1 \tag{3.7}$$

the last equation being a particular Gauss sum (see, e.g., ref. 8).

Remark 3. One might also consider interaction matrices with zero diagonal terms, recovering orthogonality in large N limit. This amounts to consider the shifted Hamiltonian

$$\tilde{\mathscr{H}}_N(\sigma) = \mathscr{H}_N(\sigma) + \tfrac{1}{2} \tag{3.8}$$

so that the average energy is equal to zero (instead of $-1/2$), and may be convenient for particular purposes (see also Remark 1).

3.1. Mean Field Properties at Any Temperature: The Second Order

The results of this section hold for the particular choice of the *sine model*

$$J_{i,j} = \frac{2}{\sqrt{2N+1}} \sin\left(\frac{2\pi ij}{2N+1}\right) \tag{3.9}$$

defined above.

Proposition 3.1. Denote $D_r(N)$ the fat diagonal of dimension r, i.e., the set of points of $\mathbf{Z}_N^r$ in which at least two of the indices coincide. Let the bar indicate the complementary set $\overline{D_r(N)}$. For every positive inverse temperature β except, at most, a set of zero Lebesgue measure, the Gibbs state $\langle\,\rangle$ expectations fulfill the following relation:

$$\Bigg| \frac{1}{N^2} \sum_{i,j,l,m \in \overline{D_4(N)}} J_{i,j} J_{l,m} \langle \sigma_i \sigma_j \sigma_l \sigma_m \rangle \\ - \frac{1}{N^2} \sum_{i,j \in \overline{D_2(N)};\, l,m \in \overline{D_2(N)}} J_{i,j} J_{l,m} \langle \sigma_i \sigma_j \rangle \langle \sigma_l \sigma_m \rangle \Bigg| = \mathscr{O}\left(\frac{1}{N}\right)$$

Remark 4. The previous formula is *homogeneous* in the following sense: thanks to restriction outside the fat diagonal, the first term only contains 4-points correlations, and the second terms only products of 2-points correlations. In this sense it can be considered as the natural generalization of the simple factorization property valid for the Curie–Weiss case.

Conceptually the proof relies on the equivalence of the microcanonical and canonical ensemble which in one of its formulations says that the energy density has vanishing fluctuations with respect to the Gibbs measure in the thermodynamical limit.

The theorem is structured in several lemmata:

Lemma 3.1. For every positive temperature β but at most a set of zero Lebesgue measure, the internal energy density has zero quadratic fluctuations, i.e.,

$$\langle h_N^2 \rangle - \langle h_N \rangle^2 = \mathcal{O}\left(\frac{1}{N}\right) \tag{3.10}$$

Proof. By the definition of free energy density

$$-\beta f_N(\beta) = \frac{1}{N} \log \sum_{\sigma} \exp^{-\beta \mathscr{H}_N(\sigma)} \tag{3.11}$$

$$\frac{d}{d\beta}(\beta f_N(\beta)) = \langle h_N \rangle \tag{3.12}$$

$$\frac{d^2}{d\beta^2}(\beta f_N(\beta)) = -N(\langle h_N^2 \rangle - \langle h_N \rangle^2) \tag{3.13}$$

The function $-\beta f_N(\beta)$ is bounded and convex with bounded derivative (the boundedness comes from the bounds on the Hamiltonian[2] due to orthogonality and convexity, and is proved on very general grounds in ref. 9). The $N \to \infty$ limit of $-\beta f_N(\beta)$, which again by convexity always exists, at least along subsequences,[9] is itself convex and has always right and left derivatives which coincide except at most on a countable set of points. Integrating the (3.13) in any β interval the positivity of the left hand side and the fundamental theorem of calculus yield the lemma.

Lemma 3.2. Case $i = j$; for the sine interaction J the following result holds:

$$\left| \frac{1}{N^2} \sum_{i,l,m} J_{i,i} J_{l,m} \langle \sigma_l \sigma_m \rangle \right| \leqslant \frac{1}{2N} \tag{3.14}$$

Proof. We have

$$\left| \frac{1}{N^2} \sum_{i,l,m} J_{i,i} J_{l,m} \langle \sigma_l \sigma_m \rangle \right| \leqslant \left| \frac{1}{N^2} \sum_{l,m} J_{l,m} \langle \sigma_l \sigma_m \rangle \right| \leqslant \frac{1}{N^2} \cdot \frac{N}{2} = \frac{1}{2N} \tag{3.15}$$

where the first inequality comes from $\sum_i J_{i,i} \leqslant 1$ which in turn is a consequence of (3.5)–(3.6). The second inequality is true because the maximum of the Hamiltonian $\sum_{l,m} J_{l,m}\sigma_l\sigma_m$, which is an upper bound for its expectation, is $N/2$.

Lemma 3.3. Case $j=l$; for the sine interaction J the following result holds:

$$\frac{1}{N^2}\sum_{i,j,m} J_{i,j}J_{j,m}\langle\sigma_i\sigma_m\rangle=\frac{1}{N} \tag{3.16}$$

Proof.

$$\frac{1}{N^2}\sum_{i,j,m} J_{i,j}J_{j,m}\langle\sigma_i\sigma_m\rangle=\frac{1}{N^2}\sum_{i,m}\delta_{i,m}\langle\sigma_i\sigma_m\rangle=\frac{1}{N^2}\sum_i 1=\frac{1}{N} \quad \blacksquare$$

Proof of Proposition 3.1. Defining

$$A=\left|\frac{1}{N^2}\sum_{i,l,m} J_{i,i}J_{l,m}\langle\sigma_l\sigma_m\rangle\right|$$

and

$$B=\left|\frac{1}{N^2}\sum_{i,j,m} J_{i,j}J_{j,m}\langle\sigma_i\sigma_m\rangle\right|$$

we have

$$\begin{aligned}&\left|\frac{1}{N^2}\sum_{i,j,l,m\in\overline{D_4(N)}} J_{i,j}J_{l,m}\langle\sigma_i\sigma_j\sigma_l\sigma_m\rangle\right.\\&\left.\quad-\frac{1}{N^2}\sum_{i,j\in\overline{D_2(N)};\,l,m\in\overline{D_2(N)}} J_{i,j}J_{l,m}\langle\sigma_i\sigma_j\rangle\langle\sigma_l\sigma_m\rangle\right|\\&\quad\leqslant\langle h_N^2\rangle-\langle h_N\rangle^2+6A+4B\end{aligned}$$

The previous lemmata provide the claim. $\blacksquare$

3.2. High Temperature Expansion of the Free Energy for the Orthogonal Model

We can now try to mimic the procedure used in the previous section to decouple the spins. To this end, let B be an orthogonal matrix such that $B^T J B = D$ with $D = \text{diag}(d_1, \ldots, d_N)$. Since $\det J \neq 0$ we have $d_i > 0$, $i = 1, \ldots, N$, and $\det J^{-1} = \prod_i d_i^{-1}$. Let $u \in \mathbb{R}^N$ be such that $\sigma = Bu$. We have $\langle J\sigma, \sigma \rangle = \langle Bu, JBu \rangle = \langle u, Du \rangle$, and thus

$$\exp\left(\frac{\lambda}{2N}\langle J\sigma, \sigma\rangle\right) = \prod_{i=1}^{N} \exp\left(\frac{\lambda}{2N} d_i u_i^2\right)$$

$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x_i^2}{2} + \sqrt{\frac{\lambda d_i}{N}}\, u_i x_i\right) dx_i$$

$$= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}\langle x, x\rangle + \sqrt{\frac{\lambda}{N}}\langle u, D^{1/2} x\rangle\right) dx$$

$$= \frac{\det J^{-\frac{1}{2}}}{(2\pi\lambda)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2\lambda}\langle y, J^{-1} y\rangle + \left\langle \sigma, \frac{y}{\sqrt{N}}\right\rangle\right) dy$$

By (2.3) this yields

$$Z_N(\beta) = 2^N \frac{\det J^{-\frac{1}{2}}}{(2\pi\beta)^{N/2}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2\beta}\langle y, J^{-1} y\rangle + \sum_i \log\cosh\frac{y_i}{\sqrt{N}}\right) dy \tag{3.17}$$

to be compared with (2.14). We point out that the square roots appearing in the above formula are only apparently ill defined. Indeed they disappear as soon as one takes its development in powers of β, because the latter contains only even terms. $Z_N(\beta)$ has been computed by Parisi and Potters in ref. 10 using standard high-temperature techniques. Relying on their computation we are now in the position to state a result analogous to Proposition 2.2 for this class of models.

Proposition 3.2. For $0 \leqslant \beta < 1$ and for any orthogonal interaction we have

$$-\beta F(\beta) \equiv \log Z_N(\beta) = N \log 2 + N G_N(\beta) \tag{3.18}$$

where

$$G_N(\beta) \nearrow G(\beta) = \frac{1}{4}\left[\sqrt{1+4\beta^2} - \log\left(\frac{1+\sqrt{1+4\beta^2}}{2}\right) - 1\right] \quad \text{as} \quad N \to \infty \tag{3.19}$$

3.3. Limiting Behaviour and Connected Correlations at High Temperature for the Orthogonal Model

We start noticing that the function $G(\beta)$ defined in Proposition 3.2 has the following expansion in the vicinity of $\beta = 0$:

$$G(\beta) = \frac{\beta^2}{4} + \mathcal{O}(\beta^3) \tag{3.20}$$

which, by the way, coincides with what one obtains for the SK model if truncated after the first term. Moreover, according to Proposition 3.2 and with the same notation of the previous section, we have

$$\langle e^{-\lambda h_N} \rangle = \frac{Z_N\left(\beta + \frac{\lambda}{N}\right)}{Z_N(\beta)} = \exp(\lambda\, G'_N(\beta^*)) \tag{3.21}$$

for some β^* such that $0 \leqslant \beta^* - \beta \leqslant \lambda/N$. We now observe that according to (3.18) we have

$$\langle h_N \rangle = -\frac{1}{N}\left(\frac{\partial \log Z_N}{\partial \beta}\right) = -G'_N(\beta) \tag{3.22}$$

and since $\langle h_N \rangle$ is bounded uniformly in N property (3.19) implies

$$\langle h_N \rangle \to \langle h \rangle = -G'(\beta) = -\frac{\beta}{1 + \sqrt{1 + 4\beta^2}} \qquad \text{as} \quad N \to \infty \tag{3.23}$$

Therefore, if we fix $\beta \in [0, 1)$ and expand the r.h.s. of (3.21) in a neighborhood of β^* we obtain for N large enough

$$\langle e^{-\lambda h_N} \rangle = e^{\lambda\, G'(\beta)}\left(1 + \mathcal{O}\left(\frac{\lambda^2}{N}\right)\right) \tag{3.24}$$

Note that $G'(0) = 0$, so that at infinite temperature ($\beta = 0$) we recover the same result as in Proposition 2.3 for this class of models (see also ref. 2, Section 3). More generally we have the following,

Proposition 3.3. For $0 \leqslant \beta < 1$ and $N \to \infty$ the energy densities h_N converge in distribution to a random variable h which is δ-distributed at $x = -G'(\beta)$.

Mimicking again the argument of the Curie–Weiss case we introduce the nth moment

$$\langle h_N^n \rangle = \frac{(-1)^n}{Z_N N^n} \left(\frac{\partial^n Z_N}{\partial \beta^n} \right)$$

$$= \frac{(-1)^n}{(2N)^n} \sum_{\substack{i_n,\dots,i_n \in \mathbf{Z}_N^n \\ j_1,\dots,j_n \in \mathbf{Z}_N^n}} J_{i_n j_1} \cdots J_{i_n j_n} \langle \sigma_{i_n} \sigma_{j_1} \cdots \sigma_{i_n} \sigma_{j_n} \rangle \tag{3.25}$$

and the nth cumulant

$$\langle h_N^n \rangle_c = \frac{(-1)^n}{N^n} \left(\frac{\partial^n \log Z_N}{\partial \beta^n} \right)$$

$$= \frac{(-1)^n}{(2N)^n} \sum_{\substack{i_n,\dots,i_n \in \mathbf{Z}_N^n \\ j_1,\dots,j_n \in \mathbf{Z}_N^n}} J_{i_n j_1} \cdots J_{i_n j_n} \langle \sigma_{i_n} \sigma_{j_1}, \dots, \sigma_{i_n} \sigma_{j_n} \rangle_c \tag{3.26}$$

where $\langle \sigma_{i_n} \sigma_{j_1}, \dots, \sigma_{i_n} \sigma_{j_n} \rangle_c$ is defined as above (see (2.32) and (2.33)). We may now use Proposition 3.3 to conclude that the cumulants vanish in the thermodynamic limit. However, at variance with the (translationally invariant) Curie–Weiss model where summing over the indices $i_1,\dots, i_n$, $j_1,\dots, j_n$ produces only an overall combinatorial factor, so that the vanishing of the nth cumulant can be immediately translated into the vanishing of pairwise n-points connected correlations, here we have to be content with the following result.

Proposition 3.4. For any orthogonal interaction and $0 \leqslant \beta < 1$ we can find a positive constant $C = C(n)$ so that as $N \to \infty$

$$\frac{1}{N^n} \sum_{\substack{i_n,\dots,i_n \in \mathbf{Z}_N^n \\ j_1,\dots,j_n \in \mathbf{Z}_N^n}} J_{i_n j_1} \cdots J_{i_n j_n} \langle \sigma_{i_n} \sigma_{j_1}, \dots, \sigma_{i_n} \sigma_{j_n} \rangle_c = \mathcal{O}\left(\frac{C}{N} \right) \tag{3.27}$$

We are now in position to strengthen, for the sine interaction, the previous proposition into a factorization-like formula by showing that for each expectation only the terms outside the *fat diagonal* contribute to the sum. For this purpose we can prove the following:

Proposition 3.5. For the sine interaction $0 \leqslant \beta < 1$ we can find a positive constant $C = C(n)$ so that as $N \to \infty$

$$\frac{1}{N^n} \sum^* J_{i_n j_1} \cdots J_{i_n j_n} \langle \sigma_{i_n} \sigma_{j_1}, \dots, \sigma_{i_n} \sigma_{j_n} \rangle_c = \mathcal{O}\left(\frac{C}{N}\right)$$

where the starred sum means the following: first apply to the pairwise n-points connected correlations the definition (2.33); then for each of the resulting terms sum only over distinct indices within each expectation.

Example. Let $n = 3$. Then:

$$\sum^* J_{i_n j_1} J_{i_2 j_2} J_{i_3 j_3} \langle \sigma_{i_n} \sigma_{j_1}, \sigma_{i_2} \sigma_{j_2}, \sigma_{i_3} \sigma_{j_3} \rangle_c$$

$$= \sum_{i_n, i_2, i_3, j_1, j_2, j_3 \in \overline{D_6(N)}} J_{i_n j_1} J_{i_2 j_2} J_{i_3 j_3} \langle \sigma_{i_n} \sigma_{j_1} \sigma_{i_2} \sigma_{j_2} \sigma_{i_3} \sigma_{j_3} \rangle$$

$$- \sum_{\substack{i_n, i_2, j_1, j_2 \in \overline{D_4(N)} \\ i_3, j_3 \in \overline{D_2(N)}}} J_{i_n j_1} J_{i_2 j_2} J_{i_3 j_3} \langle \sigma_{i_n} \sigma_{j_1} \sigma_{i_2} \sigma_{j_2} \rangle \langle \sigma_{i_3} \sigma_{j_3} \rangle$$

$$- \sum_{\substack{i_n, i_3, j_1, j_3 \in \overline{D_4(N)} \\ i_2, j_2 \in \overline{D_2(N)}}} J_{i_n j_1} J_{i_2 j_2} J_{i_3 j_3} \langle \sigma_{i_n} \sigma_{j_1} \sigma_{i_3} \sigma_{j_3} \rangle \langle \sigma_{i_2} \sigma_{j_2} \rangle$$

$$- \sum_{\substack{i_2, i_3, j_2, j_3 \in \overline{D_4(N)} \\ i_n, j_1 \in \overline{D_2(N)}}} J_{i_n j_1} J_{i_2 j_2} J_{i_3 j_3} \langle \sigma_{i_2} \sigma_{j_2} \sigma_{i_3} \sigma_{j_3} \rangle \langle \sigma_{i_n} \sigma_{j_1} \rangle$$

$$+2 \sum_{\substack{i_n, j_1 \in \overline{D_2(N)} \\ i_2, j_2 \in \overline{D_2(N)};\, i_3, j_3 \in \overline{D_2(N)}}} J_{i_n j_1} J_{i_2 j_2} J_{i_3 j_3} \langle \sigma_{i_2} \sigma_{j_2} \rangle \langle \sigma_{i_3} \sigma_{j_3} \rangle \langle \sigma_{i_n} \sigma_{j_1} \rangle$$

Proof of Proposition 3.5. The proof is by induction over $n \geqslant 2$. Set first $n = 2$. Then the proof follows from Proposition 3.4 and Lemmas 3.2 and 3.3 by the same argument of Proposition 3.1. The inductive argument then proceeds as follows: for each n Proposition 3.4 yields a quantity of order $1/N$. To prove Proposition 3.5 we have to show that the contribution to (3.27) coming from each term belonging to the fat diagonal vanishises at least as $1/N$ as $N \to \infty$. We prove this last part by induction. Since in the fat diagonal at least two indices coincide we separate two cases:

(1) The two indices belong to the same pair; for instance, $i_1 = j_1$;

(2) The two indices belong to different pairs.

Let us first consider case 1. Here the l.h.s. of (3.27) becomes

$$P_1(N) := \frac{1}{N^n} \sum_{\substack{i_n,\dots,i_n \in \mathbf{Z}_N^n \\ i_n,\dots,j_n \in \mathbf{Z}_N^n}} J_{i_n i_n} \cdots J_{i_n j_n} \langle \sigma_{i_n}\sigma_{i_n},\dots,\sigma_{i_n}\sigma_{j_n}\rangle_c$$

However $\sigma_{i_n}\sigma_{i_n} = 1$; hence the sum over i_1 can be performed. Since $\sum_i J_{ii}$ is either 0 or 1 the result is either $P_1(N) = 0$ or

$$P_1(N) = \frac{1}{N^n} \sum_{\substack{i_2,\dots,i_n \in \mathbf{Z}_N^n \\ j_1,\dots,j_n \in \mathbf{Z}_N^n}} J_{i_2 i_2} \cdots J_{i_n j_n} \langle \sigma_{i_2}\sigma_{i_2},\dots,\sigma_{i_n}\sigma_{j_n}\rangle_c$$

Therefore, in any case we get the estimate

$$|P_1(N)| \leqslant \frac{1}{N}\frac{1}{N^{n-1}} \sum_{\substack{i_2,\dots,i_n \in \mathbf{Z}_N^n \\ j_1,\dots,j_n \in \mathbf{Z}_N^n}} J_{i_2 i_2} \cdots J_{i_n j_n} \langle \sigma_{i_2}\sigma_{i_2},\dots,\sigma_{i_n}\sigma_{j_n}\rangle_c = \mathcal{O}\left(\frac{1}{N}\right)$$

because we can apply (3.27) for the $(n-1)$-points connected correlations. The same arguments applies to any other term $P_k(N)$: $k = 2,\dots, n$ where obviously $P_k(N)$ is the l.h.s. of (3.27) with $i_k = j_k$.

There remains case 2. Here there will be either terms in $\langle \sigma_{i_n}\sigma_{j_1},\dots, \sigma_{i_n}\sigma_{j_n}\rangle_c$ for which the two equal indices (for example $i_1 i_2$) appear within the same expectations or terms where they appear in different ones. For the former we observe that the identity $\sigma_{i_n}\sigma_{i_2} = 1$ reduces us to a pairwise $(n-1)$-points connected correlation. For the latter the use of (2.33) where the right hand side summation is extended only to those partitions that keep the indices i_1 and i_2 in different expectations allows us to use the inductive hypothesis (since they are just products of pairwise connected correlations of lower order). Therefore we can factor a term of order $1/N$ out of each expectation, thus concluding the proof. ∎

Example (Case n=3). Then by the formula (2.33) and $\sigma^2 = 1$:

$$\langle \sigma\sigma_{j_1}, \sigma\sigma_{j_2}, \sigma_{i_3}\sigma_{j_3}\rangle_c$$
$$= \langle \sigma_{j_1}\sigma_{j_2}, \sigma_{i_3}\sigma_{j_3}\rangle_c - \langle \sigma\sigma_{j_1}, \sigma_{i_3}\sigma_{j_3}\rangle_c \langle \sigma\sigma_{j_2}\rangle_c - \langle \sigma\sigma_{j_2}, \sigma_{i_3}\sigma_{j_3}\rangle_c \langle \sigma\sigma_{j_1}\rangle_c$$

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